

Fundamental theorem of asset pricing: a strengthened version and p -summable markets

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Abstract

In the article a strengthened version of the 'Fundamental Theorem of asset Pricing' for one-period market model is proven. The principal role in this result play total and nonannihilating cones.

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Introduction

One of the corner stones of the theory of Mathematical Finance is the so-called 'Fundamental Theorem of asset Pricing' (in fact there is a series of results under this name). The Fundamental Theorem of asset Pricing links arbitrage free markets (i.e markets that do not admit riskless claims yielding profit with strictly positive probability; the accurate definition will be given below in Section 1) with existence of martingales generated by measures that are equivalent to the initial one. One should mention quite a number of researchers who contributed to the theme. Among them are F. Black, M. Scholes, R. Merton, J. Harrison, S. Pliska, S. Ross, D.M. Kreps, R. Dalang, A. Morton, W. Willinger, D. Kramkov, J. Jacod, A.N. Shiryaev, F. Delbaen, W. Schachermayer and many others. We cannot give a full account of sources and names related to the subject and refer, for example, to [1], [2], and [3] and the sources quoted therein.

In the article [4] the authors described geometry of Banach structures forming mathematical base of the 'Fundamental Theorem of asset Pricing' type phenomena for the one-period financial market model. One of the main results of [4] is recalled in Theorem 1.4 of the present article. There are two principal assumptions in this theorem: the first one is nonemptiness of the interior of the cone $\overset{\circ}{K}^* \neq \emptyset$, where K is the cone of arbitrage possibilities (profit cone), and the second one is reflexivity of the subspace $L = L^{**}$, where L is the subspace of financial market strategies. Both these conditions are trivially satisfied in the standard situation usually considered in the 'Fundamental Theorem of asset Pricing' (cf. Theorems 1.2 and 1.3 of the present article), namely, $K = L_{1+}$ is the

cone of nonnegative functions in $L_1(\Omega, P)$ and L is a finite dimensional subspace. As is shown in [4], Theorem 4.1 once one relaxes essentially conditions on K (for example, do not presupposes satisfaction of the property $\overset{\circ}{K}^* \neq \emptyset$) martingality somehow disappears from arbitrage freeness criterium: in a general situation there is no condition like condition 2) in Theorem 1.4 and only a certain analog of condition 3) takes place. Moreover, arbitrage freeness criterium itself reduces to the classical theorem on bipolar (see the proof of Theorem 4.1, [4]). One of the goals of the present article is the analysis of relaxation of conditions on K without loosing the martingality nature of arbitrage free markets. We will find out that the principal condition under which the martingality nature of arbitrage free markets is still alive is existence of total and nonanihilating cones in K^* . Such cones can radically reduce the 'search region' for martingale measures. As is shown by concrete examples this region can be essentially smaller than $\overset{\circ}{K}^*$ and in fact often exists even when $\overset{\circ}{K}^* = \emptyset$. In addition in a number of situations this observation makes it possible to strengthen Theorems 1.2, 1.3, and 1.4.

The article is organized as follows. In the starting Section 1 we recall a one-period market model and the corresponding Fundamental Theorem of asset Pricing. Then we give a geometric reformulation of this theorem (Theorem 1.2) and formulate its refined version (Theorem 1.3) which, in fact, is a result from [4]. Here we also give a general description of arbitrage free markets geometry (Theorem 1.4; [4], Theorem 2.4). The next Section 2 presents the principal results of the article. We introduce total and nonanihilating cones and give examples of these objects. The main result (Theorem 2.3) is a strengthened version of the Fundamental Theorem of asset Pricing exploiting objects in question. As a particular corollary of this result and examples considered we obtain the corresponding Fundamental Theorem of asset Pricing for p -summable markets (Theorem 2.4). We finish the article with a comment on a finite dimensional situation where we find out that the set of objects in question reduces to a single element (Theorems 2.5 and 2.6).

1 Arbitrage free markets geometry

In this section we recall the Fundamental Theorem of asset Pricing for one-period market model and discuss its geometric nature. It will be the starting point of our further analysis.

A one-period market model is given in the following way. Let us denote by $\bar{\pi} := (\pi^0, \pi) := (\pi^0, \pi^1, \dots, \pi^d) \in \mathbb{R}_+^{d+1}$ the (initial, known) price system at moment t_0 . By $\bar{S} := (S^0, S) := (S^0, S^1, \dots, S^d)$ we denote the price system at moment t_1 , that is a family of nonnegative random variables on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ (where $(\Omega, \mathcal{F}, \mathcal{P})$ is the space of (possible) *scenario*). It is assumed that all the random variables under consideration are summable, that is $S^i \in L_1(\Omega, P)$, $i = \overline{0, d}$. In addition the variable S^0 is assumed to be a riskless bond, that is it is not random

$$S^0 := (1 + r) \pi^0, \quad (1.1)$$

where r is interpreted as a bank interest rate (for purely mathematical reasons one can assume that $r > -1$). In what follows we presuppose that

$$\pi^0 = 1, \quad (1.2)$$

that is the price π^0 is normalized. Therefore

$$S^0 := (1 + r); \quad (1.3)$$

A (starting) investment *portfolio* is a vector $\bar{\xi} := (\xi^0, \xi) := (\xi^0, \xi^1, \dots, \xi^d) \in \mathbb{R}^{d+1}$, where the values ξ^i can be negative.

The price of buying the portfolio (at moment t_0) is equal to

$$\bar{\xi} \cdot \bar{\pi} := \sum_{i=0}^d \xi^i \pi^i. \quad (1.4)$$

And the value of portfolio (at moment t_1) is the random variable

$$\bar{\xi} \cdot \bar{S} = \sum_{i=0}^d \xi^i S^i(\omega) = \xi^0 (1 + r) + \xi \cdot S. \quad (1.5)$$

An *arbitrage opportunity* is a portfolio $\bar{\xi} \in \mathbb{R}^{d+1}$, such that

$$\bar{\xi} \cdot \bar{\pi} \leq 0, \text{ but } \{\bar{\xi} \cdot \bar{S} \geq_{\text{a.e.}} 0 \text{ and } P(\bar{\xi} \cdot \bar{S} > 0) > 0\}.$$

If the market is arbitrage free (i.e. there are no portfolio satisfying the relation written above) it is reasonable to consider it as being just.

It is convenient to express the arbitrage freeness conditions in terms of the so-called discounted net gains. Recall that *discounted net gains* (at moment t_1) are the random variables given by

$$Y^i := \frac{S^i}{1+r} - \pi^i, \quad i = 1, \dots, d. \quad (1.6)$$

Let us denote by Y the vector of discounted net gains $Y := (Y^1, \dots, Y^d)$.

By (1.1) we have $Y^0 = \frac{S^0}{1+r} - \pi^0 = 0$ and therefore Y^0 does not play any role.

Lemma 1.1. [[5], Lemma 1.3 and condition (1.3)] *The following conditions are equivalent:*

- 1) *market is arbitrage free;*
- 2) *if $\xi \in \mathbb{R}^d$ satisfies $\xi \cdot Y \geq_{\text{a.e.}} 0$, then $\xi \cdot Y =_{\text{a.e.}} 0$.*

This lemma has clear geometric interpretation.

Let

$$L := \{\xi \cdot Y, \xi \in \mathbb{R}^d\} = \left\{ \sum_{i=1}^d \xi_i Y^i, (\xi_1, \dots, \xi_d) \in \mathbb{R}^d \right\} \quad (1.7)$$

be the subspace generated by the vectors (functions) Y^i , $i = 1, \dots, d$. By L_{1+} we denote the cone of nonnegative functions

$$L_{1+} := \{f \in L_1(\Omega, P), f \geq 0\}. \quad (1.8)$$

Lemma 1.1 means that

$$\text{a market is arbitrage free} \Leftrightarrow L \cap L_{1+} = \{0\}. \quad (1.9)$$

The foregoing observations make it natural to consider L_{1+} as the cone of *arbitrage possibilities* (*profit cone*) and consider L as the subspace of *financial market strategies*.

Evidently it is important to obtain description of (geometric, algebraic and etc.) conditions under which the equality $L \cap L_{1+} = \{0\}$ takes place. The most known market arbitrage freeness condition in financial mathematics is the *fundamental theorem of asset pricing*. It sounds as follows (see, for example, [5], Theorem 1.6):

a market is arbitrage free \Leftrightarrow there exists a martingale measure P^* which is equivalent to the initial measure P and has a bounded Radon-Nikodim derivative $\frac{dP^*}{dP}$.

In this article we prove a certain strengthened version of this result (Theorem 2.3). To implement this we need a geometric reformulation of the above mentioned fundamental theorem of asset pricing. Hereafter we present it.

Let us consider a Banach space $L_1(\Omega, P)$. As usually, elements of this space are equivalence classes of integrable functions, where the equivalence of two functions is given by their equality almost everywhere; and the norm is given by the integral. Thus, all the equalities and inequalities are understood as 'almost everywhere'.

As is known, for the dual space $L_1(\Omega, P)^*$ we have $L_1(\Omega, P)^* = L_\infty(\Omega, P)$ (where $L_\infty(\Omega, P)$ is the Banach space of equivalence classes of essentially bounded functions with essup-norm). In this case elements $x^* \in L_\infty(\Omega, P)$ are identified with functionals (elements of $L_1(\Omega, P)^*$) by means of coupling

$$\langle x^*, u \rangle = \int_{\Omega} u x^* dP, \quad u \in L_1(\Omega, P). \quad (1.10)$$

Let us consider now the cone L_{1+} (1.8) of nonnegative functions in $L_1(\Omega, P)$. By $L_{1+}^* \subset L_1(\Omega, P)^* = L_\infty(\Omega, P)$ we denote the cone of nonnegative functionals on L_{1+} , i.e.

$$L_{1+}^* := \{x^* \in L_\infty(\Omega, P) : \langle x^*, u \rangle \geq 0 \text{ for every } u \in L_{1+}\}. \quad (1.11)$$

Evidently, L_{1+}^* coincides with the cone $L_{\infty+}$ of nonnegative functions from $L_\infty(\Omega, P)$, i.e.

$$L_{1+}^* = L_{\infty+} := \{x^* \in L_\infty(\Omega, P), x^* \geq 0\}. \quad (1.12)$$

We denote by $\tilde{L}_{\infty+}$ the cone

$$\tilde{L}_{\infty+} := \{x^* \in L_\infty(\Omega, P), x^* >_{\text{a.e.}} 0\}. \quad (1.13)$$

By $L^\perp \subset L_1(\Omega, P)^*$ we denote the subspace of functionals annihilating on L (L is the subspace (1.7) generated by the vectors Y^i , $i = 1, \dots, d$).

As is shown in [4] one can rewrite the mentioned fundamental theorem of asset pricing in the following way.

Theorem 1.2. [fundamental theorem of asset pricing: geometric formulation: [4], Theorem 1.3]

For the objects described above the following two conditions are equivalent:

- 1) $L \cap L_{1+} = \{0\}$ (= absence of arbitrage);
- 2) $L^\perp \cap \tilde{L}_{\infty+} \neq \emptyset$ (= existence of a martingale measure).

In [4], in particular, a certain 'refined version' of this result is proven. Namely, let us consider the cone

$$\overset{\circ}{L}_{\infty+} = \{x^* \in L_{\infty+}, x^* \text{ essentially separated from zero } 0\}. \quad (1.14)$$

Clearly, $\overset{\circ}{L}_{\infty+} \subset \tilde{L}_{\infty+}$ and $\overset{\circ}{L}_{\infty+}$ is nothing else as the interior of the cone $L_{\infty+}$ (1.12).

Theorem 1.3. [fundamental theorem of asset pricing: 'refined version'; [4], Theorem 1.3]

For the objects described above the following two conditions are equivalent:

- 1) $L \cap L_{1+} = \{0\}$ (= absence of arbitrage);
- 2) $L^\perp \cap \overset{\circ}{L}_{\infty+} \neq \emptyset$ (= existence of a martingale measure (refined condition)).

Moreover, the main goal of [4] was to uncover a general geometric nature of arbitrage freeness type phenomena. The corresponding geometric picture is given by the next

Theorem 1.4. [arbitrage free markets geometry; [4], Theorem 2.4] *Let E be a Banach space, $K \subset E$ be a closed cone such that $\overset{\circ}{K}^* \neq \emptyset$ and $L \subset E$ be a closed linear reflexive subspace $L = L^{**}$. For the objects mentioned above the following two conditions are equivalent:*

- 1) $L \cap K = \{0\}$ (= absence of arbitrage);
- 2) $L^\perp \cap \overset{\circ}{K}^* \neq \emptyset$ (= existence of a martingale measure);
- 3) $L^\perp + \overset{\circ}{K}^* = E^*$, where E^* is the dual space to E .

As we have noted in Introduction, once one relaxes essentially conditions on K (for example, do not presupposes satisfaction of the property $\overset{\circ}{K}^* \neq \emptyset$) martingality somehow disappears from arbitrage freeness criterium: in a general situation there is no condition like condition 2) in Theorem 1.4 and only a certain analog of condition 3) takes place (see [4], Theorem 4.1).

The main results of the article are presented in the next section, where we undertake the analysis of relaxation of conditions on K without losing the martingality nature of arbitrage free markets. In addition we will find out that in a number of situations Theorems 1.2, 1.3, and 1.4 can be strengthened in a way.

2 Fundamental theorem of asset pricing: a strengthened version

To formulate the results we need to introduce a number of objects.

Let E be a Banach space and $K \subset E$ be a certain cone. Recall that a *cone* in a vector space is a set K possessing the following two properties:

- 1) K is a convex set;
- 2) for every $x \in K$ and any $0 < \lambda \in \mathbb{R}$ one has $\lambda x \in K$.

As usually by K^* we denote the cone of nonnegative functionals on K , i.e.

$$K^* := \{x^* \in E^* : \langle x^*, u \rangle \geq 0 \text{ for any } u \in K\}, \quad (2.1)$$

here E^* is the space dual to E .

Let $K = \overline{K}$ be a closed cone. A cone $\mathcal{K} \subset K^*$ is called *total* if it possesses the following property: let $u \in E$ and for every $x^* \in \mathcal{K}$ one has $\langle x^*, u \rangle \geq 0$, then $u \in K$. The set of all total cones in K^* will be denoted by \mathbf{K}_{tot}^* .

Examples 1. Clearly the cone K^* itself is total.

2. If K^* has a nonempty interior $\overset{\circ}{K}^* \neq \emptyset$, then $\overset{\circ}{K}^*$ is a total cone. Indeed, in this case we have $\overline{\overset{\circ}{K}^*} = K^*$. If $u \notin K$, then there exists a functional $x^* \in K^*$, such that $\langle x^*, u \rangle < 0$. Therefore for functionals $x^{*'} \in \overset{\circ}{K}^*$ that are sufficiently close to x^* one has $\langle x^{*'}, u \rangle < 0$ as well.

3. In particular, if $L_{\infty+}$ is the cone (1.12) of nonnegative functions in $L_\infty(\Omega, P)$, which is interpreted as the cone L_{1+}^* of nonnegative functionals on the cone L_{1+} of nonnegative functions in $L_1(\Omega, P)$ (1.8), then the cone

$$\overset{\circ}{L}_{\infty+} = \{Q \in L_{1+}^* (= L_{\infty+}), Q \text{ is essentially separated from zero}\}. \quad (2.2)$$

is a total cone.

4. For the cone $L_{1+}^* = L_{\infty+}$ (1.12) let us consider one more cone $\tilde{L}_{\infty+}$ (1.13). Clearly,

$$\langle Q, f \rangle \geq 0 \text{ for every } Q \in \tilde{L}_{\infty+} \Leftrightarrow f \in L_{1+}, \quad (2.3)$$

that is $\tilde{L}_{\infty+}$ is a total cone as well.

5. Let again $L_{1+}^* = L_{\infty+}$ be the cone (1.12), and \mathcal{K} be the cone of simple positive functions having finite sets of values, i.e. finite sums of the form

$$f = \sum_i c_i \chi(A_i),$$

where $\Omega \supset A_i$ are measurable sets, $\coprod_i A_i = \Omega$ and $c_i > 0$.

Clearly \mathcal{K} is a total cone.

A cone $\mathcal{K} \subset K^*$ is said to be *nonannihilating* if it possesses the following property: for every $x^* \in \mathcal{K}$ and any $0 \neq u \in K$ one has $\langle x^*, u \rangle > 0$. The set of all nonannihilating cones in K^* is denoted by \mathbf{K}_{nan}^* .

Examples. 6. If K^* has a nonempty interior $\overset{\circ}{K}^* \neq \emptyset$, then $\overset{\circ}{K}^*$ is a nonannihilating cone. Indeed, suppose that $x^* \in \overset{\circ}{K}^*$ and $\langle x^*, u \rangle = 0$. In this case let us take any functional $y^* \in E^*$ such that $\langle y^*, u \rangle = \alpha < 0$. Since $x^* \in \overset{\circ}{K}^*$ it follows that for sufficiently small positive ε one has $x^* + \varepsilon y^* \in \overset{\circ}{K}^*$ and therefore $\langle x^* + \varepsilon y^*, u \rangle = \varepsilon \alpha < 0$. Thus we arrived at a contradiction.

7. In particular, if $L_{1+}^* = L_{\infty+}$ is the cone (1.12) then the cone $\tilde{L}_{\infty+}$ (2.2) is nonannihilating.

8. For the same cone $L_{1+}^* = L_{\infty+}$ (1.12) the cone $\tilde{L}_{\infty+}$ (1.13) is nonannihilating as well.

9. Let again $L_{1+}^* = L_{\infty+}$ be the cone (1.12) and \mathcal{K} be the cone of simple positive functions having finite sets of values. Clearly, \mathcal{K} is a nonannihilating cone.

10. Unfortunately not for every cone K one has $\mathbf{K}_{nan}^* \neq \emptyset$. Let, for example, $K = L$ be a linear subspace. In this situation $L^* = L^\perp$. Thus, for every $x^* \in L^*$ and for any $u \in L$ we have $\langle x^*, u \rangle = 0$.

The first essential result in description of arbitrage free markets is the following observation.

Lemma 2.1. *Let E be a Banach space and $K, L \subset E$, where K is a cone and L is a linear subspace. For every nonannihilating $\mathcal{K} \in \mathbf{K}_{nan}^*$ the following relation is true*

$$L^\perp \cap \mathcal{K} \neq \emptyset \Rightarrow L \cap K = \{0\}. \quad (2.4)$$

Proof. Implication (2.4) is equivalent to the implication

$$L \cap K \neq \{0\} \Rightarrow L^\perp \cap \mathcal{K} = \emptyset.$$

Therefore we prove the latter one.

Let

$$0 \neq u_0 \in L \cap K.$$

Consider the functional $\bar{u}_0 \in E^{**}$ given by relation

$$\bar{u}_0(x^*) := \langle x^*, u_0 \rangle, \quad x^* \in E^*.$$

To prove the equality $L^\perp \cap \mathcal{K} = \emptyset$ it is enough to check that the functional \bar{u}_0 strictly separates L^\perp and \mathcal{K} , that is

$$\text{for every } x^* \in L^\perp \quad \langle x^*, u_0 \rangle = 0, \quad (2.5)$$

$$\text{for every } x^* \in \mathcal{K} \quad \langle x^*, u_0 \rangle > 0. \quad (2.6)$$

Equality (2.5) follows from the fact that $u_0 \in L$, and inequality (2.6) is true since $\mathcal{K} \in \mathbf{K}_{nan}^*$. The lemma is proved.

The next observation is presented by

Lemma 2.2. *Let E be a Banach space and $K, L \subset E$, where $K = \overline{K}$ is a closed cone, and L is a finite-dimensional linear subspace. For every total cone $\mathcal{K} \in \mathbf{K}_{tot}^*$ the following implication is true*

$$L \cap K = \{0\} \Rightarrow L^\perp \cap \mathcal{K} \neq \emptyset. \quad (2.7)$$

Proof. Let us introduce the set

$$C := \{x^*|_L, x^* \in \mathcal{K}\} \subset L^*, \quad (2.8)$$

that is the set of restrictions of functionals \mathcal{K} onto L (here L^* is the dual space to L , that is the set of linear continuous functionals on L). By assumption L is finite-dimensional. Therefore, $L \cong L^*$.

Evidently,

$$L^\perp \cap \mathcal{K} \neq \emptyset \Leftrightarrow C \ni 0, \quad (2.9)$$

where $0 \in L^*$.

Relation (2.9) shows that to prove (2.7) it is enough to verify the following implication

$$0 \notin C \Rightarrow L \cap K \neq \{0\}. \quad (2.10)$$

Therefore, hereafter we prove (2.10).

Since \mathcal{K} is a cone we have that C is a cone in L^* .

As C is a cone and $0 \notin C$ it follows (by separation theorem) that there exists $f_0 \in L$, such that

- i) for every $x^* \in \mathcal{K} \quad \langle x^*, f_0 \rangle \geq 0$;
- ii) there exists $x_0^* \in \mathcal{K}$, such that $\langle x_0^*, f_0 \rangle > 0$.

Since $\mathcal{K} \in \mathbf{K}_{tot}^*$ condition i) implies $f_0 \in K$; and condition ii) implies $f_0 \neq 0$. So, (2.10) is verified and the proof is complete.

Combining Lemmas 2.1 and 2.2 we obtain

Theorem 2.3. [fundamental theorem of asset pricing: a strengthened version] *Let E be a Banach space and $K, L \subset E$, where $K = \overline{K}$ is a closed cone, and L is a finite-dimensional linear subspace. Suppose that $\mathbf{K}_{tot}^* \cap \mathbf{K}_{nan}^* \neq \emptyset$. Then for every cone $\mathcal{K} \in \mathbf{K}_{tot}^* \cap \mathbf{K}_{nan}^*$ the following to conditions are equivalent:*

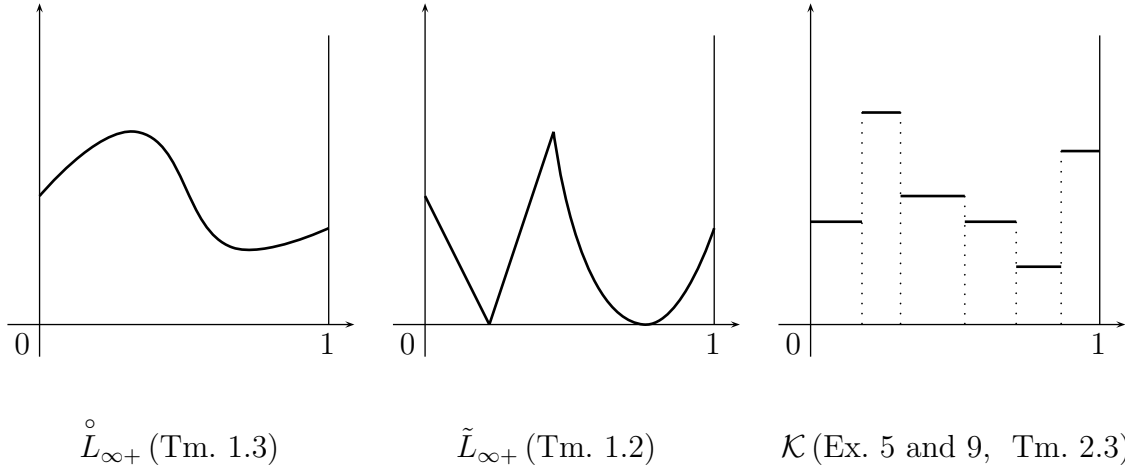
- 1) $L \cap K = \{0\}$ (= absence of arbitrage);
- 2) $L^\perp \cap \mathcal{K} \neq \emptyset$ (= existence of a martingale measure (a strengthened condition)).

Note that the above presented Examples 2 and 6 show that if the cone K^* has a nonempty interior $\overset{\circ}{K}^* \neq \emptyset$, then $\overset{\circ}{K}^* \in \mathbf{K}_{tot}^* \cap \mathbf{K}_{nan}^*$. Thus Theorem 2.3 for a finite-dimensional subspace L , in particular, implies equivalence of 1) and 2) in Theorem 1.4.

The next figure illustrates the difference between Theorems 1.2, 1.3 and 2.3 by means of the cone \mathcal{K} from Examples 5 and 9. Note that in a general situation one has the following relations for the cones $\tilde{L}_{\infty+}$ (Theorem 1.2), $\overset{\circ}{L}_{\infty+}$ (Theorem 1.3) and \mathcal{K} (Examples 5 and 9):

$$\tilde{L}_{\infty+} \supsetneq \overset{\circ}{L}_{\infty+} \supsetneq \mathcal{K}$$

and therefore Theorem 2.3 is the strongest assertion among the mentioned ones.



One has to emphasize that property $K^* \neq \emptyset$ is rather special (see in this connection a discussion of this property in [4], Section 3). This property takes place for the cone $L_{1+}^* = L_{\infty+}$ (1.12) (and namely this cone is exploited in Theorem 1.2). On the other hand, if, for example, one considers the analogous cones L_{p+} of nonnegative functions in the spaces $L_p(\Omega, P)$, $1 \leq p < \infty$, then here one has $\overset{\circ}{L}_{p+} = \emptyset$. In addition for $1 \leq p < \infty$ we have $L_{p+}^* = L_{q+}$, where $\frac{1}{q} + \frac{1}{p} = 1$ and we assume $\frac{1}{\infty} = 0$. Let us note, however, that for every cone $K := L_{p+}$, $1 \leq p < \infty$ one again has $\mathbf{K}_{tot}^* \cap \mathbf{K}_{nan}^* \neq \emptyset$. Indeed, for example, the following three cones are elements of this set

- a) $\mathcal{K}_1 = \{Q \in L_{q+}, Q \geq_{\text{a.e.}} 0\}$;
- b) $\mathcal{K}_2 = \{Q \in \mathcal{K}_1, Q \text{ is essentially separated from zero}\}$;
- c) \mathcal{K}_3 is the cone of simple positive functions having finite sets of values.

Therefore, Theorem 2.3, in particular, implies the next result, which is reasonable to consider as a strengthened version of the fundamental theorem of asset pricing for p -summable markets.

Theorem 2.4. [fundamental theorem of asset pricing for p -summable markets] *Let $K := L_{p+}$ be the cone of nonnegative functions in $L_p(\Omega, P)$, $1 \leq p < \infty$ and $L \subset L_p(\Omega, P)$ be a finite-dimensional linear subspace. Then for every of the cones \mathcal{K}_i , $i = 1, 2, 3$ mentioned above the following two conditions are equivalent:*

- 1) $L \cap K = \{0\}$ (= absence of arbitrage);
- 2) $L^\perp \cap \mathcal{K}_i \neq \emptyset$ (= existence of a martingale measure).

To finish the article we give two comments on the subspace finite dimensionality condition. First of all note that in the situation $E = \mathbb{R}^n$ the set $\mathbf{K}_{tot}^* \cap \mathbf{K}_{nan}^*$ reduces to a single element.

Theorem 2.5. *If $K \subset \mathbb{R}^n$ is a closed cone such that $K \cap (-K) = 0$ then $\overset{\circ}{K}^* \neq \emptyset$ and*

$$\mathbf{K}_{tot}^* \cap \mathbf{K}_{nan}^* = \{\overset{\circ}{K}^*\} \neq \emptyset.$$

Proof. Note that under the conditions of theorem K is a plasterable cone (see [4], Example 3.3.) and we have $\overset{\circ}{K}^* \neq \emptyset$. Thus (see Examples 2 and 6)

$$\overset{\circ}{K}^* \in \mathbf{K}_{tot}^* \cap \mathbf{K}_{nan}^*,$$

and, in particular, $\mathbf{K}_{tot}^* \cap \mathbf{K}_{nan}^* \neq \emptyset$.

Now let us consider any element $\mathcal{K} \in \mathbf{K}_{tot}^* \cap \mathbf{K}_{nan}^*$. We have to prove that $\mathcal{K} = \overset{\circ}{K}^*$. Since K is plasterable [4], Theorem 3.1., 5) implies that

$$F := \overline{co}(K \cap \{u : \|u\| = 1\}) \not\supset 0,$$

where $co(A)$ is the convex hull of A .

As $\mathcal{K} \in \mathbf{K}_{nan}^*$ one has that

$$\langle x^*, u \rangle > 0 \text{ for every } x^* \in \mathcal{K} \text{ and every } u \in F. \quad (2.11)$$

Since F is compact (2.11) implies that for every $x^* \in \mathcal{K}$ there exists a constant $c > 0$ such that

$$\langle x^*, u \rangle > c \text{ for every } u \in F. \quad (2.12)$$

Therefore

$$\langle x^*, u \rangle > c\|u\| \text{ for every } 0 \neq u \in K.$$

Which means that $x^* \in \overset{\circ}{K}^*$ (by [4], Theorem 3.1., 3)). So we have proved the inclusion

$$\mathcal{K} \subset \overset{\circ}{K}^*.$$

Now let us prove the opposite inclusion.

First of all note that

$$\overline{\mathcal{K}} = K^*. \quad (2.13)$$

The proof goes by contradiction. Suppose that

$$\overline{\mathcal{K}} \neq K^*.$$

It means that there exists $y^* \in K^*$ such that $y^* \notin \overline{\mathcal{K}}$. By separation theorem we conclude that there exists $u \in \mathbb{R}^n$ such that

$$\langle x^*, u \rangle \geq 0 \text{ for every } x^* \in \mathcal{K}, \quad (2.14)$$

and

$$\langle y^*, u \rangle < 0. \quad (2.15)$$

Since $\mathcal{K} \in \mathbf{K}_{tot}^*$ it follows that (2.14) implies $u \in K$ which contradicts (2.15). Thus (2.13) is proved.

Now take any $y^* \in \overset{\circ}{K}^*$. By Caratheodory's Theorem there exists $\{y_1^*, \dots, y_{n+1}^*\} \subset K^*$ such that

$$y^* \in co(\{y_1^*, \dots, y_{n+1}^*\}) \quad (2.16)$$

Bearing in mind (2.13) and taking $\{x_1^*, \dots, x_{n+1}^*\} \subset \mathcal{K}$ in such a way that each x_i^* , $i = 1, \dots, n+1$ is sufficiently close to the corresponding y_i^* we obtain from (2.16) that

$$y^* \in co(\{x_1^*, \dots, x_{n+1}^*\}).$$

Thus $\overset{\circ}{K}^* \subset \mathcal{K}$ and the proof is finished.

As a corollary of the theorem just proved we conclude that in the situation under consideration Theorem 2.3 reduces to

Theorem 2.6. Consider the space \mathbb{R}^n . Let $K \subset \mathbb{R}^n$ be a closed cone such that $K \cap (-K) = \{0\}$. For a linear subspace $L \subset \mathbb{R}^n$ the following two conditions are equivalent:

- 1) $L \cap K = \{0\}$,
- 2) $L^\perp \cap \overset{\circ}{K}^* \neq \emptyset$, where L^\perp the orthogonal complement to L .

Our final observation shows that in general finite dimensionality condition for the subspace L is essential in formulation of the strengthened version of the fundamental theorem of asset pricing (if the dimension of L is infinite then not for every $\mathcal{K} \in \mathbf{K}_{tot}^* \cap \mathbf{K}_{nan}^*$ properties 1) and 2) of Theorem 2.3 are equivalent).

Example. Let $E = l_1 = \{(\xi_1, \xi_2, \dots) : \sum_i |\xi_i| < \infty\}$ and let $K \subset l_1$ be the cone of nonnegative sequences and L be the subspace generated by vectors of the form $e_{2n} - \frac{1}{2n}e_{2n-1}$, $n = 1, 2, \dots$, where $e_k, k = 1, 2, \dots$ is the canonical base in l_1 . Evidently L is nothing else than the space of vectors of the form $\sum_{n=1}^\infty \xi_n(e_{2n} - \frac{1}{2n}e_{2n-1}) : \sum_{n=1}^\infty |\xi_n| < \infty$. Clearly

$$K \cap L = \{0\}.$$

The dual space to l_1 is the space l_∞ of bounded sequences and the action of an element $f = (\nu_1, \nu_2, \dots) \in l_\infty$ on $x = (\xi_1, \xi_2, \dots) \in l_1$ is given by the coupling $f(x) = \sum_i \xi_i \nu_i$. In this example $K^* \subset l_\infty$ is the cone of nonnegative bounded sequences. Let $\mathcal{K} = \overset{\circ}{K}^*$ be the set of positive sequences separated from zero (clearly $\overset{\circ}{K}^* \in \mathbf{K}_{tot}^* \cap \mathbf{K}_{nan}^*$). Evidently, if $f = (\nu_1, \nu_2, \dots) \in L^\perp$ then $\nu_{2n-1} = \frac{1}{2n}\nu_{2n}$. This along with the boundness of the sequence (ν_{2n}) implies $\nu_{2n-1} \rightarrow 0$. Therefore $f \notin \mathcal{K}$, that is

$$L^\perp \cap \mathcal{K} = \emptyset.$$

Thus in this example $\mathbf{K}_{tot}^* \cap \mathbf{K}_{nan}^* \neq \emptyset$ and here condition 1) of Theorem 2.3 is satisfied while condition 2) is not satisfied.

To finish the article we note that the material presented naturally causes the problem: can we obtain a general description of the cones K possessing the property $\mathbf{K}_{tot}^* \cap \mathbf{K}_{nan}^* \neq \emptyset$?

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